The Development of Students' Use of Justification Strategies

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This paper examines young students' development and use of justification strategies when engaged in numeric patterning activities. Drawing on findings from a three-month teaching experiment aimed to improve students' early algebraic understanding we show how student use of justification strategies can be extended through the use of specifically designed tasks and pedagogical actions. In particular, we examine how students' ability to participate in mathematical argumentation can be supported through use of justification which triangulates numeric, verbal and visual strategies.

Introduction

Difficulties in developing student understanding of algebraic concepts are well-documented in New Zealand and internationally (Irwin & Britt, 2005; Knuth, Stephens, McNeil, & Alibabi, 2006). A key difficulty identified within recent research studies relates to the need for students to justify generalisations using arguments which support their development of mathematical proof (Carpenter, Levi, Berman, & Pligge, 2005). Many researchers (e.g., Blanton & Kaput, 2005; Carraher, Schliemann, Brizuela, & Earnest, 2006) argue that student participation in practices of conjecturing, generalising, and justifying mathematical reasoning are fundamental to the development of foundations of algebraic reasoning. The recent New Zealand curriculum document advocates that students learn to "justify and verify, and to seek patterns and generalisations" (Ministry of Education (MoE), 2007, p. 26). However, developing such practices in mathematics classrooms is challenging for many teachers and students, particularly because they may not have previously experienced classrooms in which justification of mathematical reasoning takes a central role. The research reported in this paper examines how a classroom environment and the teachers' pedagogical actions facilitate primary school students to use a range of justifications to develop generalisations of functional patterns.

Teachers take a significant role in scaffolding the specific questions and prompts which move students from explaining their solution strategies to justifying, defending, and generalising their solution strategies (Hunter, 2007). Justification is a critical element of the generalisation process. However, determining the legitimacy of a general statement is a demanding task for young students (Lannin, 2005). Most often, studies have found that elementary students initially view specific examples or trying a number of cases as valid justification (Carpenter et al., 2005; Lannin). Research has also shown, however, that in classrooms where students have opportunities to participate in mathematical argumentation and justification, the quality of students' reasoning, explanations, and justification can be enhanced (Manoucheri & St John, 2006; McCrone, 2005; Wood, Williams, & McNeal, 2006). Through establishing classroom social and socio-mathematical norms which require students to justify through triangulation of verbal, numerical and graphical strategies, students can be supported to learn ways to provide further, more sophisticated forms of justification (Kazemi, 1998).

Within this process, research has revealed the importance of students learning to make connections across representations. In particular, the use of visual and numeric patterns has been found to support students to identify, communicate, and justify functional rules. For example, Carpenter and his colleagues illustrated in a 6th grade classroom that students could be scaffolded to provide concrete justification through the use of materials. Other studies (e.g., Beatty & Moss, 2006; Healy & Hoyles, 1999) highlight the importance of teachers drawing student attention to the visual representations of functional patterns. This, coupled with a teacher press for students to communicate and justify their generalisations using the geometric context, resulted in a more robust student understanding of functions and ability to find, express, and justify functional rules.

In our study, drawing on the emergent perspective of Cobb (1995), students' mathematical learning is recognised as both an individual constructive process and as social negotiation of meaning; neither is given more significance than the other. Combining both Piagetian and Vygotskian notions of cognitive development the person, cultural, and social factors are all viewed as important features within the students' learning environment. From this theoretical perspective we attempt "to offer a developing picture of what it looks like for a teacher's practice to cultivate students' algebraic reasoning skills in robust ways" (Blanton & Kaput, 2005, p. 440) through specific focus on the development of justification strategies.

Method

The findings reported in this paper are one section of a larger study (Hunter, 2007) which involved a 3-month classroom teaching experiment (Cobb, 2000). The research was conducted at a New Zealand urban primary school. The 25 participant students were between 9 and 11 years old. The student group came from a predominantly middle socio-economic home environment and included a range of ethnic backgrounds.

All students participated in pre- and post-interviews. Working as collaborative partners, the researcher and teacher used data gathered from the pre-interview to develop an initial sequence of learning activities informed by a hypothetical learning trajectory focused on developing early algebraic understanding. The learning trajectory was used to establish learning activities involving tasks and participatory practices and associated learning climate that required students to make conjectures, justifications, and generalisations. The initial tasks and activities focused students on exploration of the properties of number and associated computations. The students were then provided with problems designed to develop algebraic reasoning through the use of linear functional problems and patterning activities including tasks with a geometric context.

Throughout the teaching experiment data were generated and collected through participant observations, video records and classroom artefacts. Subsequent on-going data analysis and collaborative examination of classroom practices by the researcher and teacher led to modification of the instructional sequence. Findings of the one classroom case study were based on retrospective data analysis that used a grounded approach identifying categories, codes, patterns, and themes.

Results and Discussion

Before we focus on the nature of the teaching experiment with regard to developing students' understanding of functional relationships, we first present a summary of the pre- and post-test results of participants' understanding of functional relationships. Table 1 overviews the number of students able to use a functional relationship[3] at the first interview conducted at the start of the classroom study.

Table 1

Percentage of Students (n=25) Correctly Using the Functional Relationship

	Correct response	Incorrect response	No response
Part A	40%	52%	8%
Part B	28%	52%	20%

Comparable data from the final interview (see Table 2) indicate that there was an increase in the number of students able to correctly use the functional relationship to solve the problem—thus demonstrating impact in terms of performance measures.

Table 2

Percentage of Students (n=25) Correctly Using the Functional Relationship

	Correct response	Incorrect response	No response
Part A	88%	12%	0%
Part B	84%	12%	4%

The focus of this paper is to explore one aspect of how such learning was occasioned. Tasks were designed

3 1) To make copies of a CD, a store charges a set-up fee and an additional amount per CD. The store charges $2 as a set-up fee and an additional $3 for each copy.

A) What is the cost to make 10 copies of a CD?

B) What is the cost to make 21 copies of a CD?

and selected to promote students' algebraic reasoning through the explicit use of numerical, verbal, and visual schema and to support their development of justification. Of equal importance to the design of the tasks were the planned pedagogical interventions. In the next section, we consider the pedagogical practices—involving teacher scaffolding, modelling, and a press for all students to interact and justify their reasoning through argumentation—evidenced in the study.

Developing Justification Beyond Numeric Examples

Initially, many students maintained a view that using lots of different numbers justified their reasoning. For example, when challenged by the teacher Hayden explained:

> Hayden: We tested it with this number, it works, this number, it works and we are just finishing this one and so far it is working.

To shift students beyond the use of multiple examples the teacher required that they integrate visual and numeric schema and justify their generalisations using the geometric problem representation. In a large group discussion Hamish explained his group's generalisation with reference to the problem context:

> Hamish: Thirty-two people sit at the table … you get the ten and times it by three and the two people who are sitting on those ends, one of them stays there and the other one gets moved to the end of the new table.

The teacher pressed him to justify his contextual explanation using the geometric pattern. In doing this, she demonstrated how to justify the link between the functional rule and the geometric representation through modelling (see Figure 1).

> Teacher: Hamish can you show … the times three part of your model there and the plus two part? … This is the first table here so I'm going to use blue for the times three and the plus two here [uses pink counters on each end] … so two tables, two groups of three do you see that? [Points to blue counters.] Plus two [points to pink counters].

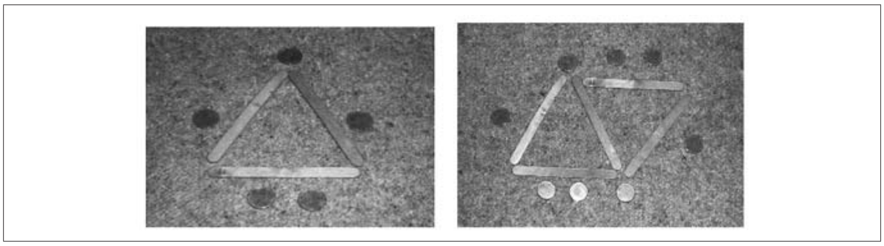

Figure 1. Using a geometric representation to justify a functional rule.

Positioning Students to Justify Algebraic Reasoning

The teacher modelled how participants were required to justify their reasoning. She explicitly positioned the students to take a stance to explain and justify their reasoning and provided space for participants to agree or disagree mathematically. For example, in a large group discussion Mike stated an incorrect generalisation for a problem[4]:

> Mike: You would times it by five then you would minus one because of the six … it would be one over so you would have to minus this to make it fair.

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4 Jasmine and Cameron are playing "Happy houses". They have to build a house and add onto it. The first one looks like this….. / \
|_|
The second building project looks like this… / \ / \
|_|_|
How many sticks would you need to build four houses? How many sticks would you need to build eight houses? Can you find a pattern and a rule?

The teacher revoiced but did not evaluate the conjecture:

Teacher: [revoices] Mike said the number of houses times five minus one because of the six you get at the start. Does everyone agree? Could someone show us why or why not you agree?

The teacher, by asking the students to take a position to agree or disagree and provide convincing evidence of their stance, effectively increased the press for justification. Subsequently, a student argued her position through drawing a house that acted as a referent for her generalisation:

Ruby: [draws a house] That is one house and if you added another one, that is always going to be a six when you times it by five you would actually add one because you have timesed that by five and it's still a six so you would add it on.

This provided an avenue for the teacher to step in as a participant and provide a counter argument that modelled how to justify a position using equipment:

Teacher: [builds representation of two houses] I could show you another way why it doesn't work. Now I have to times by five and two times five is ten, now if I take it away I am going to have an incomplete house. I have to add one so that is my two times five, to make it complete I need to add one.

Through teacher modelling, a sustained press to explain and justify reasoning, and explicit space provided to rethink conjectures, the students recognised that they were required to not only explain and justify their reasoning but also to be prepared to validate it. This was exemplified when a group of students recognised an emerging pattern and one member challenged:

Susan: We have got to make proof that it actually works because if you think it is eleven but you don't know that it is eleven.

Increasingly, as the teaching sequence progressed, students demonstrated awareness of the need for explanatory justification to extend beyond the provision of multiple examples. This was further illustrated in a lesson when a student argued that justification required more than multiple examples:

Ruby: It's not really saying anything … do you think it would be more convincing if we used equipment [for our explanation]?

The argument presented by Ruby pressed other group members to shift towards connecting the visual pattern and functional generalisation.

Triangulating Numeric, Verbal, and Visual Strategies to Justify Functional Generalisations

The previous exemplars illustrated that concrete materials, as well as inscriptions of mathematical reasoning, were important thinking and communicating tools. The teacher requirement that the students triangulate their numerical, verbal, and graphical strategies appeared to become an accepted practice—a practice that facilitated more sophisticated and shared understanding of the justification process. For example, she required that they link their use of a t-chart with visual representations when justifying their explanations:

Teacher: You need to be able to show and prove that by drawing a table and using equipment.

On several occasions the teacher was also observed emphasising the importance of constructing links between the numeric pattern in the t-chart and the geometric models. During student provided explanation she frequently drew attention to how the explainer had linked the numeric and geometric patterns as illustrated in the following response:

Teacher: This is really important this part, what he is going to show you will help you make the links between that and the table of data.

The combination of the use of the designed linear functional problems involving geometric patterns and the teacher press prompted students to use equipment to support their justifying processes. Furthermore, requiring students to link numeric and geometric patterns encouraged them to construct reasoned explanations which incorporated multiple representations. This is illustrated in a large group discussion when Ruby justifies her explicit generalisation of the house problem using a geometric pattern:

Ruby: [builds model] The first one is six but then when you add another house it is only five because you don't need another wall … if you wanted to see how many for eight you could just go eight times five and then plus the one, you are plusing the one because you have to still understand that that is six [points to first house].

Again in a later lesson, Josie and Matthew repeatedly refer to the geometric context of the problem to convince their other group members of their explicit generalisation and to justify their reasoning. When other group members remain unconvinced, they use multiple forms to justify their reasoning. After examining the model (see Figure 2), Josie begins by pointing with her hands to the already drawn model:

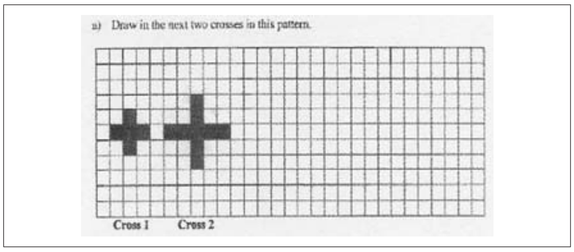

Figure 2. Cross problem.

Josie: This is cross one. There is one on each side plus one in the middle. This is cross two, so two here and two here and one in the middle so that makes five. So you double it and then add one to get the number across.

Matthew builds on it using a visualised representation to explain how to find the number of squares across for cross twenty:

Matthew: [Indicates a vertical line with his hands] It would be twenty and twenty plus one so that would be forty-one.

However, the other group members Steve and Rani remain unconvinced and continue to press for justification:

Steve: So why does it equal forty-one?

Josie and Matthew again refer to the already drawn model to provide additional justification for their explicit generalisation:

Matthew: [points to each side] Because like cross one has one there and one there.

Josie: Plus one in the middle.

Matthew: Two has two there and two there and one in the middle and three has three there and three there plus one.

The questioning stance taken by other group members prompts Matthew to redraw two models using different colours (see Figure 3):

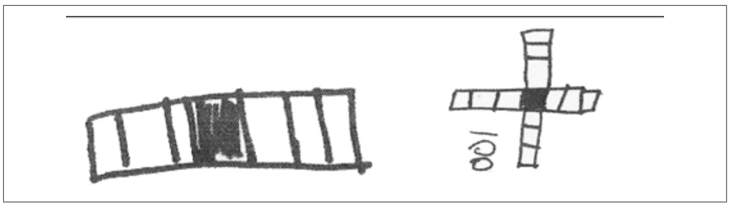

Figure 3. Matthew's justification for his generalisation.

Matthew: One, two, three and the one in the middle is black. So this is for cross three. Right, have you noticed a pattern yet?

Steve: No I haven't.

Matthew: Cross three has got three, cross two has got two on the outside.

Josie: Each part that sticks out has the amount, the number of the cross then there is one in the middle to join it up … it is the number of the cross doubled and then you add one in the middle to get that amount there [covers the model so only the horizontal row is showing].

The participants pause to examine the explanation; however Steve and Rani recognise their right to question until convinced. They reframe their questions as they analyse the reasoning:

Steve: So when you double it, what are you actually trying to get to by doubling it? ...

Josie: [covers the vertical row so only the horizontal row is visible] The number of squares in that line there.

Rani looks closely at the inscription and then she revoices the generalisation using the model of the cross to clarify her understanding.

Rani: [points to one arm of the cross] so you double that and add one.

Josie: You double the number of the cross and add one.

Rani: [points to the arms of the cross] So, you double these?

Josie: [points to right horizontal arm] That little bit here [points to upwards vertical arm] this little bit here is also three squares wide and this is three squares wide [points to left horizontal arm] and that is three squares wide [points to downwards vertical arm] so to get the bit across here in the middle [points to the horizontal row] you do times two plus one.

At Josie's statement all the students nod in agreement. Through the extended analytical discussion the students illustrated that they recognised the need to determine the legitimacy of the generalised reasoning beyond immediate justification.

Conclusion and Implications

This study sought to explore how students were supported to use justification to develop rich understandings of early algebraic reasoning. In particular, focus was placed on how students can be scaffolded to use justification strategies which triangulate numeric, verbal, and visual strategies. Similar to the findings of other researchers (e.g., Carpenter et al., 2005; Lannin, 2005), many of the students initially viewed trying a number of cases as valid justification. Specially designed tasks coupled with the purposeful teacher actions led to students developing understanding of the need to justify beyond examples and determine the legitimacy of the justification.

As other researchers have previously described (Manoucheri & St John, 2006; McCrone, 2005; Wood et al., 2006), opportunities to engage in argumentation and justification enhanced both the quality of students' reasoning, and their ability to determine the validity of a justification. The small sample of emblematic learning activities which were used in the teaching experiment demonstrate how the activities involving functional problems with a geometric base, in combination with specific pedagogical actions, can successfully extend student communication of justification strategies. The task design involving geometric patterns and use of multiple representations provided many opportunities for students to explore, justify, and validate their reasoning using equipment. A further pedagogical press from the teacher including questioning and positioning of students and the requirement that they take a stance led to them justifying their reasoning through an increased use of concrete materials, inscriptions, and geometric representations.

Implications of this study suggest that a triangulated approach of teacher press for justification, task design, and use of materials scaffold students to develop multiple forms of justification. However, further research is required to validate the findings of this study due to the small sample of participants involved.

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